

POSITIVE SOLUTIONS FOR THE FRACTIONAL LAPLACIAN IN THE ALMOST CRITICAL CASE IN A BOUNDED DOMAIN

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ABSTRACT. We prove existence of multiple positive solutions for a *fractional scalar field equation* in a bounded domain, whenever p tends to the critical Sobolev exponent. By means of the “photography method”, we prove that the topology of the domain furnishes a lower bound on the number of positive solutions.

1. INTRODUCTION

In the celebrated papers [1, 2] Benci, Cerami and Passaseo proved an existence result of positive solutions of the following problem

$$(1.1) \quad \begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $N \geq 3$ and $p < 2^* = \frac{2N}{N-2}$, the critical Sobolev exponent of the embedding of $H_0^1(\Omega)$ in the Lebesgue spaces. Roughly speaking they show that (among other results), for p near 2^* , the number of positive solutions is bounded below by a topological invariant associated to Ω . More specifically they prove the following

Theorem. *There exists a $\bar{p} \in (2, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ problem (1.1) has (at least) $\text{cat}_{\bar{\Omega}}(\bar{\Omega})$ positive solutions. Even more, if Ω is not contractible in itself, the number of solutions is $\text{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$.*

Hereafter given a topological pair $A \subset X$, $\text{cat}_X(A)$ is the Ljusternik-Schnirelmann category of the set A in X (see e.g. [6]).

To prove this result, the authors used variational methods: an energy functional related to the problem is introduced in such a way that the solutions are seen as critical point of this functional restricted to L^p -ball. Then the “photography method” (which permits to see a photography of the domain Ω in a suitable sublevel of the functional) is implemented in order to prove the existence of many critical points by means of the classical Ljusternick-Schnirelmann Theory.

The aim of this paper is to prove the fractional counterpart of the above Theorem. Indeed, due to the large literature appearing in these last years on fractional operators, it is very natural to ask if a similar result also holds for the fractional laplacian. In other words we consider in this paper the following nonlocal problem

$$(1.2) \quad \begin{cases} (-\Delta)^s u + u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$, $p \in (2, 2_s^*)$ with $2_s^* := 2N/(N - 2s)$, $N > 2s$.

The operator $(-\Delta)^s$ is the *fractional Laplacian* which is defined by

$$(-\Delta)^s u(x) := C(N, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

2010 *Mathematics Subject Classification.* 35A15, 55M30, 58E05.

Key words and phrases. Fractional Laplacian, Variational Methods, Ljusternick-Schnirelmann category, multiplicity of solutions.

Giovany M. Figueiredo was partially supported by CNPq, Brazil. Gaetano Siciliano was partially supported by Fapesp and CNPq, Brazil.

for a suitable constant $C(N, s) > 0$ whose exact value is not really important for our purpose. The Dirichlet condition in (1.2) is then given on $\mathbb{R}^N \setminus \Omega$ reflecting the fact that $(-\Delta)^s$ is a nonlocal operator.

Before to state our result, let us introduce few basic notations. For a measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ let

$$[u]_{D^{s,2}(\mathbb{R}^N)}^2 := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

be the (squared) *Gagliardo seminorm* of u . Let us define the Hilbert space

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) : [u]_{D^{s,2}(\mathbb{R}^N)}^2 < +\infty \right\},$$

which is continuously embedded into $L^{2^*}(\mathbb{R}^N)$. Let finally

$$D_0^{s,2}(\Omega) = \left\{ u \in D^{s,2}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

From now on it will be convenient to adopt the following convention: functions defined in a subset of \mathbb{R}^N , let us say A , will be thought extended by zero on $\mathbb{R}^N \setminus A$, whenever regarded as functions defined on the whole \mathbb{R}^N .

Note that being $\partial\Omega$ smooth, $D_0^{s,2}(\Omega)$ can be also defined as the completion of $C_0^\infty(\Omega)$ under the norm $[\cdot]_{D^{s,2}(\mathbb{R}^N)}$. Moreover, it is $D_0^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$.

Recall that we have the continuous embedding $D_0^{s,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 2_s^*$ and that the embedding is compact for $1 \leq p < 2_s^*$.

We then say that $u \in D_0^{s,2}(\Omega)$ is a solution (in the distributional sense) of (1.2) if

$$(1.3) \quad \forall v \in D_0^{s,2}(\Omega) : \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} u v dx = \int_{\mathbb{R}^N} |u|^{p-2} u v dx.$$

The main result of the paper gives a positive answer on the possibility of extending the Benci, Cerami and Passaseo result to the fractional case.

Theorem 1. *For $s \in (0, 1)$, $N > 2s$, there exists a $\bar{p} \in (2, 2_s^*)$ such that for every $p \in [\bar{p}, 2_s^*)$ problem (1.2) possesses (at least) $\text{cat}_{\bar{\Omega}}(\bar{\Omega})$ positive solutions. Whenever Ω is not contractible in itself, the number of solutions is $\text{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$.*

Beside the case when $p \rightarrow 2^*$, the papers [1, 2] treat also when a certain parameter λ appearing in the equation tends to $+\infty$. We do not enter in details here, but the same type of result is obtained: for large λ the domain topology gives a lower bound on the number of positive solutions of the problem. However it is readily seen that this last case can be equivalently reformulated as a problem in an expanding domain, simply by a change of variables which “transfer” the parameter λ from the equation to the domain. In the same spirit, in [3] the influence of the domain topology is studied for semiclassical equations, that is, roughly speaking, when a parameter ε which appears in the equation tends to zero.

Subsequently, after the papers [1–3], many authors have used the same methods to prove multiplicity results of solutions (depending on the domain topology) whenever $\lambda \rightarrow +\infty$, that is for problems in expanding domains, or $\varepsilon \rightarrow 0$, that is for semiclassical states.

Nevertheless, to the best of our knowledge, there is only another paper in the literature dealing with the case in which the role of parameter is taken by the exponent of the nonlinearity which tends to the critical Sobolev exponent: see [8] where the Schrödinger-Poisson system is studied. We think that, even if less explored, the case in which the parameter is the exponent of the power nonlinearity is equally interesting; our goal is then to give a contribution in this direction. Observe finally, that by considering $s = 1$, our proof can be adapted to recover the result of [1] by using the method of the Nehari manifold, in place of the L^p -ball as done in the paper of Benci and Cerami.

The paper is organized in the following way.

In Section 2 we give the variational setting in which problem (1.2) is settled. Section 3 deals with a related limit problem, which will be useful in order to prove Theorem 1. Finally, in Section 4 after introducing the barycenter map and prove some important properties, the proof of Theorem 1 is given.

Let us finish this section with basic notations that will be used in all the paper.

Notations. Without loss of generality we assume in all the paper $0 \in \Omega$. We denote by $|\cdot|_{L^p(A)}$ the L^p -norm of a function defined on the domain A . If the domain is Ω or \mathbb{R}^N (it should be clear from the context) we will use the notation $|\cdot|_p$.

We use $B_r(y)$ for the closed ball of radius $r > 0$ centered in y . If $y = 0$ we simply write B_r .

The letter c will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero.

Other notations will be introduced whenever we need.

2. THE VARIATIONAL SETTING

It is easily seen that a solution in the sense (1.3) of problem (1.2) can be found as a critical point of the C^1 functional

$$(2.1) \quad I_p(u) = \frac{1}{2}[u]_{D^{s,2}(\mathbb{R}^N)}^2 + \frac{1}{2}|u|_2^2 - \frac{1}{p}|u|_p^p \quad u \in D_0^{s,2}(\Omega).$$

Observe that, for $u \in D_0^{s,2}(\Omega)$ we can write equivalently

$$[u]_{D^{s,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

where of course $\Omega^c := \mathbb{R}^N \setminus \Omega$. We will use the next result

Lemma 1. [9, Lemma 6] *If $u \in D_0^{s,2}(\Omega)$ then*

$$|u|_{L^{2^*_s}(\Omega)} \leq c [u]_{D^{s,2}(\mathbb{R}^N)}^2$$

for a suitable constant $c > 0$. In particular it follows that

$$\|u\|^2 := [u]_{D^{s,2}(\mathbb{R}^N)}^2 + |u|_2^2$$

gives an equivalent (squared) norm on $D_0^{s,2}(\Omega)$.

Then we can write

$$I_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p.$$

A fundamental tool in order to apply variational techniques is the so-called *Palais-Smale condition* (PS for brevity). If M is a smooth manifold in $D_0^{s,2}(\Omega)$, we say that I_p satisfies the PS condition on M (or restricted to M) if every sequence $\{u_n\} \subset M$ such that

$$(2.2) \quad \{I_p(u_n)\} \text{ is bounded and } I'_p(u_n) \rightarrow 0 \text{ in } D^{-s,2}(\Omega),$$

admits a converging subsequence. Clearly $I'_p(u_n)$, has to be intended as the tangencial component of $I'_p(u_n)$ to $T_{u_n}M$. Sequences which satisfy (2.2) are called *Palais-Smale sequences*.

To prove the theorem we use the general ideas of Benci, Cerami and Passaseo adapting their arguments to our problem which contains a nonlocal operator.

A natural way of finding the critical points of I_p which is unbounded above and below, is to restrict the functional to a suitable manifold, *the Nehari manifold*, on which it results bounded below and hence the classical Ljusternick-Schnirelmann Theory can be employed.

2.1. The Nehari manifold. In this subsection we recall some known facts about the Nehari manifold that will be used throughout the paper.

The Nehari manifold associated to (2.1) is defined by

$$\mathcal{N}_p = \left\{ u \in D_0^{s,2}(\Omega) \setminus \{0\} : G_p(u) = 0 \right\}$$

where

$$G_p(u) := I'_p(u)[u] = [u]_{D^{s,2}(\mathbb{R}^N)}^2 + |u|_2^2 - |u|_p^p.$$

Note that on \mathcal{N}_p the functional (2.1) takes the form

$$(2.3) \quad I_p(u) = \frac{p-2}{2p}\|u\|^2 \geq 0.$$

Sometimes we will refer to (2.3) as the constraint functional, also denoted with $I_p|_{\mathcal{N}_p}$. In the next Lemma we list the basic properties of the Nehari manifold, easy to check in a standard way.

Lemma 2. *For $p \in (2, 2_s^*]$, we have*

1. \mathcal{N}_p is a C^1 manifold,
2. there exists $c > 0$ such that for every $u \in \mathcal{N}_p : c \leq \|u\|$,
3. for any $u \neq 0$ there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_p$ and $\inf_{0 \neq u \in D_0^{s,2}(\Omega)} t_u > 0$,
4. the following equalities are true

$$m_p := \inf_{u \neq 0} \max_{t > 0} I_p(tu) = \inf_{g \in \Gamma_p} \max_{t \in [0,1]} I_p(g(t)) = \inf_{u \in \mathcal{N}_p} I_p(u) > 0$$

where

$$\Gamma_p = \left\{ g \in C([0,1]; D_0^{s,2}(\Omega)) : g(0) = 0, I_p(g(1)) \leq 0, g(1) \neq 0 \right\}.$$

Moreover the manifold \mathcal{N}_p is a *natural constraint* for I_p in the sense that any critical point of I_p restricted to \mathcal{N}_p is also a critical point for the “free” functional I_p in the whole Hilbert space. Hence the (constraint) critical points we find are solutions of our problem since no Lagrange multipliers will appear.

With a standard proof, one show that the Nehari manifold well-behaves with respect to the PS sequences, that is

Lemma 3. *Let $\{u_n\} \subset \mathcal{N}_p$ be a PS sequence for $I_p|_{\mathcal{N}_p}$. Then it is a PS sequence for the free functional I_p on the whole space $D_0^{s,2}(\Omega)$. Moreover, if $p \in (2, 2_s^*)$ then I_p restricted to \mathcal{N}_p satisfies the PS condition.*

As a consequence we set

$$\forall p \in (2, 2_s^*) : m_p := \min_{\mathcal{N}_p} I_p = I_p(u_p),$$

i.e. m_p is achieved on a function (also called *ground state*), hereafter denoted with $u_p \in \mathcal{N}_p$. Observe that the family of minimizers $\{u_p\}_{p \in (2, 2_s^*)}$ is bounded away from zero; indeed, since $u_p \in \mathcal{N}_p$,

$$(2.4) \quad \|u_p\|^2 \leq |u_p|_p^p \leq C \|u_p\|^p$$

where the positive constant C is independent of p . Hence

$$\exists c > 0 \quad \text{s.t.} \quad \forall p \in (2, 2_s^*) : 0 < c \leq \|u_p\|.$$

Remark 1. *By (2.4), we deduce that $\{|u_p|_p\}_{p \in (2, 2_s^*)}$ is also bounded away from zero. Moreover, the Hölder inequality implies*

$$|u_p|_p \leq |\Omega|^{\frac{2_s^* - p}{2_s^* p}} |u_p|_{2_s^*}$$

so that also $\{|u_p|_{2_s^*}\}_{p \in (2, 2_s^*)}$ is far away from zero.

It will be important for us, in order to prove the main Theorem (see subsection 4.2), to evaluate the limit of the ground state levels m_p for p tending to 2_s^* .

3. THE LIMIT PROBLEM

With the aim of evaluating the limit of the sequence $\{m_p\}_{p \in (2, 2_s^*)}$ when $p \rightarrow 2_s^*$, we start by considering a limit problem related to (1.2). Let us introduce the C^1 functional on $D_0^{s,2}(\Omega)$

$$(3.1) \quad I_*(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2_s^*} |u|_{2_s^*}^{2_s^*}$$

whose critical points are the solutions of

$$(3.2) \quad \begin{cases} (-\Delta)^s u + u = |u|^{2_s^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The lack of compactness of the embedding of $D_0^{s,2}(\Omega)$ in $L^{2_s^*}(\Omega)$ implies that I_* does not satisfies the PS condition at every level; indeed the *fractional conformal scaling*

$$u(\cdot) \longmapsto u_R(\cdot) := R^{(N-2s)/2} u(R(\cdot)), \quad R > 1$$

leaves invariant the $[u]_{D^{s,2}(\mathbb{R}^N)}$ and the $L^{2_s^*}$ -norm of $u : \Omega \rightarrow \mathbb{R}$:

$$[u_R]_{D^{s,2}(\mathbb{R}^N)}^2 = [u]_{D^{s,2}(\mathbb{R}^N)}^2 \quad |u_R|_{2_s^*}^{2_s^*} = |u|_{2_s^*}^{2_s^*}.$$

On the other hand, for $p \in [1, 2_s^*)$,

$$|u_R|_p^p = R^{\frac{p(N-2s)-2N}{2}} |u|_p^p \longrightarrow 0 \quad \text{as } R \longrightarrow +\infty$$

and then it is easy to see that in a bounded domain Ω the infimum

$$S := \inf_{0 \neq u \in D_0^{s,2}(\Omega)} \frac{\|u\|^2}{|u|_{2_s^*}^2}$$

is never achieved. Let also

$$\mathcal{N}_* = \{u \in D_0^{s,2}(\Omega) : G_*(u) = 0\}, \quad G_*(u) = \|u\|^2 - |u|_{2_s^*}^{2_s^*}$$

be the Nehari manifold associated to problem (3.2) and

$$m_* := \inf_{\mathcal{N}_*} I_* = \inf_{u \neq 0} \max_{t > 0} I_*(tu).$$

If $u \in \mathcal{N}_*$ then $I_*(u) = \frac{s}{N} \|u\|^2$.

The following lemma is probably known but for the sake of completeness we give the proof.

Lemma 4. *There holds*

$$m_* = \frac{s}{N} S^{N/2s}$$

where S is the best (fractional) Sobolev constant defined above.

Proof. For $A, B > 0$ it results

$$\max_{t > 0} \left\{ \frac{t^2}{2} A - \frac{t^{2_s^*}}{2_s^*} B \right\} = \frac{s}{N} \left(\frac{A}{B^{2/2_s^*}} \right)^{N/2s}.$$

Then

$$m_* = \inf_{u \neq 0} \max_{t > 0} I_*(tu) = \frac{s}{N} \left(\inf_{u \neq 0} \frac{\|u\|^2}{|u|_{2_s^*}^2} \right)^{N/2s} = \frac{s}{N} S^{N/2s}.$$

□

In particular it is easy to see that m_* is not achieved.

As a first step, we show that m_* is an upper bound for the sequence of ground states levels $\{m_p\}_{p \in (2, 2_s^*)}$.

Lemma 5. *We have*

$$\limsup_{p \rightarrow 2_s^*} m_p \leq m_*.$$

Proof. Given $\varepsilon > 0$, there exists $u \in \mathcal{N}_*$ such that

$$(3.3) \quad I_*(u) = \frac{s}{N} \|u\|^2 < m_* + \varepsilon.$$

Now consider, for any $p \in (2, 2_s^*)$, the unique positive value t_p such that $t_p u_R \in \mathcal{N}_p$. By definition, t_p satisfies

$$(3.4) \quad \|t_p u_R\|^2 = |t_p u_R|_p^p$$

from which we deduce:

- $\{t_p\}_{p \in (2, 2_s^*)}$ is bounded away from zero.

Indeed by (3.4) and the embedding of $D_0^{s,2}(\Omega)$ in $L^p(\Omega)$ we get $\|t_p u_R\|^2 \leq C \|t_p u_R\|^p$ so $\|t_p u_R\|^2 \geq c$ and finally $t_p^2 \geq \frac{c}{\|u_R\|^2} \geq \frac{c}{\|u\|^2} > 0$.

- $\{t_p\}_{p \in (2, 2_s^*)}$ is bounded above.

Indeed

$$\|u_R\|^2 = t_p^{p-2} |u_R|_p^p$$

and, by the continuity of the map $p \mapsto |u_R|_p$, it is readily seen that if t_p tends to $+\infty$ as $p \rightarrow 2_s^*$ we get a contradiction.

So we may assume that $\lim_{p \rightarrow 2_s^*} t_p = t_*$, and passing to the limit in (3.4) we get

$$t_*^2 [u]_{D^{s,2}(\mathbb{R}^N)}^2 + \frac{t_*^2}{R^{2s}} |u|_2^2 = t_*^{2s} |u|_{2_s^*}^{2s} = t_*^{2s} \|u\|^2$$

or equivalently,

$$(t_*^{2s} - t_*^2) [u]_{D^{s,2}(\mathbb{R}^N)}^2 = (\frac{t_*^2}{R^{2s}} - t_*^{2s}) |u|_2^2.$$

Now if R is chosen sufficiently large, the r.h.s. above is negative and we deduce

$$(3.5) \quad t_* < 1.$$

Furthermore

$$I_p(t_p u_R) = \frac{p-2}{2p} \|t_p u_R\|^2 = \frac{p-2}{2p} t_p^2 [u]_{D^{s,2}(\mathbb{R}^N)}^2 + \frac{p-2}{2p} \frac{t_p^2}{R^{2s}} |u|_2^2$$

and passing to the limit for $p \rightarrow 2_s^*$, taking advantage of (3.5),

$$\lim_{p \rightarrow 2_s^*} I_p(t_p u_R) = \frac{s}{N} t_*^2 [u]_{D^{s,2}(\mathbb{R}^N)}^2 + \frac{s}{N} \frac{t_*^2}{R^{2s}} |u|_2^2 < \frac{s}{N} \|u\|^2$$

Lastly we get, using (3.3),

$$\limsup_{p \rightarrow 2_s^*} m_p \leq \lim_{p \rightarrow 2_s^*} I_p(t_p u_R) < \frac{s}{N} \|u\|^2 < m_* + \varepsilon$$

which concludes the proof since ε is arbitrary. \square

Recalling (2.3), the boundedness of $\{m_p\}_{p \in (2, 2_s^*)}$ implies the boundedness of the ground state solutions, namely

$$(3.6) \quad \exists c > 0 \quad \text{such that} \quad \forall p \in (2, 2_s^*) : \|u_p\| \leq c.$$

We need now another technical lemma.

Lemma 6. *For every $p \in (2, 2_s^*)$ let $w_p \in \mathcal{N}_p$ be an arbitrary function and assume that the family $\{w_p\}_{p \in (2, 2_s^*)}$ is bounded in $D_0^{s,2}(\Omega)$ and $\{|w_p|_{2_s^*}^{2s}\}_{p \in (2, 2_s^*)}$ is bounded away from zero.*

Denote with $t_p > 0$ the unique value such that $t_p w_p \in \mathcal{N}_$. Then*

$$\limsup_{p \rightarrow 2_s^*} t_p \leq 1$$

and $\{t_p\}_{p \in (2, 2_s^)}$ is bounded away from zero. In particular by (3.6) and Remark 1 the result follows for the sequence of ground state solutions $\{u_p\}_{p \in (2, 2_s^*)}$.*

Proof. By definition of \mathcal{N}_* , t_p satisfies

$$t_p^{2s} |w_p|_{2_s^*}^{2s} = t_p^2 \|w_p\|^2$$

and using that $w_p \in \mathcal{N}_p$ and the Hölder inequality we get

$$(3.7) \quad t_p^{2s-2} = \frac{|w_p|_p^p}{|w_p|_{2_s^*}^{2s}} \leq \frac{|\Omega|^{\frac{2_s^*-p}{2_s^*}}}{|w_p|_{2_s^*}^{2s-p}}.$$

By the embedding $D_0^{s,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ we deduce that the sequence $\{|w_p|_{2_s^*}^{2s}\}_{p \in (2, 2_s^*)}$ is bounded. Since

it is also bounded away from zero, the conclusion follows by (3.7), since $\lim_{p \rightarrow 2_s^*} \frac{|\Omega|^{\frac{2_s^*-p}{2_s^*}}}{|w_p|_{2_s^*}^{2s-p}} = 1$. \square

Now we can give the main result of this section.

Proposition 1. *For any bounded domain we have*

$$\lim_{p \rightarrow 2_s^*} m_p = m_*.$$

Proof. By Lemma 5 it is sufficient to prove that

$$m_* \leq \liminf_{p \rightarrow 2_s^*} m_p.$$

Let $t_p > 0$ the unique value such that $t_p u_p \in \mathcal{N}_*$; hence by Lemma 6

$$m_* \leq I_*(t_p u_p) = \frac{s}{N} t_p^2 \|u_p\|^2 = I_p(u_p) t_p^2 + \left(\frac{1}{p} - \frac{1}{2_s^*} \right) \|u_p\|^2 t_p^2 = m_p t_p^2 + o(1)$$

where $o(1) \rightarrow 0$ for $p \rightarrow 2_s^*$, and the conclusion follows. \square

The following “global compactness” result is an extension of the Struwe result (see Theorem 3.1 of [10]) to the fractional case. Its proof can be found in [7]; see also [4] for the non hilbertian case.

Theorem 2. *Let $\{v_n\}$ be a PS sequence for I_* (defined in (3.1)) in $D_0^{s,2}(\Omega)$. Then there exist a number $k \in \mathbb{N}_0$, sequences of points $\{x_n^j\} \subset \Omega$ and sequences of radii $\{R_n^j\}$ ($1 \leq j \leq k$) with $R_n^j \rightarrow +\infty$ for $n \rightarrow +\infty$, there exist a solution $v \in D_0^{s,2}(\Omega)$ of (3.2) and non trivial solutions $\{v^j\}_{j=1,\dots,k} \subset D^{s,2}(\mathbb{R}^N)$ of*

$$(3.8) \quad (-\Delta)^s u = |u|^{2_s^*-2} \quad \text{in } \mathbb{R}^N,$$

such that, a (relabelled) subsequence $\{v_n\}$ satisfies

$$\begin{aligned} v_n - v - \sum_{j=1}^k v_{R_n^j}^j(\cdot - x_n^j) &\rightarrow 0 \quad \text{in } D^{s,2}(\mathbb{R}^N), \\ I_*(v_n) &\rightarrow I_*(v) + \sum_{j=1}^k \hat{I}(v^j) \end{aligned}$$

where $\hat{I} : D^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ is given by

$$\hat{I}(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} dx.$$

Roughly speaking, the solutions of (3.8) are responsible for the lack of compactness.

For what concerns \hat{I} , it is known (see [5]) that it achieves its minimum on functions of type

$$(3.9) \quad U_R(x - a) = C_{N,s} \left(\frac{R}{R^2 + |x - a|^2} \right)^{\frac{N-2s}{2}} \quad R > 0, \quad a \in \mathbb{R}^N$$

and $C_{N,s} > 0$ is a suitable constant. The minimum value is exactly

$$\hat{I}(U_R(\cdot - a)) = \frac{N}{s} \int_{\mathbb{R}^N} |U|^{2_s^*} dx = m_*,$$

namely the infimum of I_* defined in (3.1) on \mathcal{N}_* . The value of \hat{I} on solutions of (3.8) which do not belong to the family (3.9) (which are the unique positive solutions) is greater than $2m_*$ (see [4, Lemma 2.10] for details). As a consequence, if the sequence $\{v_n\}$ of Theorem 2 is a PS sequence for I_* at level m_* , we deduce $I_*(v) = 0$, $k = 1$ and $v^1 = U$. Moreover, being v a solution of (3.2) and being I_* positive on the solutions, it follows necessarily that $v = 0$; so Theorem 2 gives

$$v_n - U_{R_n}(\cdot - x_n) \rightarrow 0 \quad \text{in } D^{s,2}(\mathbb{R}^N).$$

This “decomposition” of the Palais-Smale sequence will be fundamental in the next Section.

4. PROOF OF THEOREM 1

From now on, let $r > 0$ be a radius sufficiently small such that $B_r \subset \Omega$ (recall $0 \in \Omega$) and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^3 : d(x, \Omega) \leq r\}$$

$$\Omega_r^- = \{x \in \Omega : d(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω . Let also

$$(4.1) \quad h : \Omega_r^+ \rightarrow \Omega_r^-$$

the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity.

4.1. The barycenter map. We are in a position now to prove the main result of this section.

For $u \in D_0^{s,2}(\mathbb{R}^N)$ with compact support, let us denote with the same symbol u its trivial extension out of $\text{supp } u$. The barycenter of u is defined as

$$\beta(u) = \frac{\int_{\mathbb{R}^N} x |u|^{2_s^*}}{\int_{\mathbb{R}^N} |u|^{2_s^*}}.$$

We have the following.

Proposition 2. *There exists $\varepsilon > 0$ such that if $p \in (2_s^* - \varepsilon, 2_s^*)$, we have*

$$u \in \mathcal{N}_p \text{ and } I_p(u) < m_p + \varepsilon \implies \beta(u) \in \Omega_r^+.$$

Proof. We argue by contradiction. Assume that there exist sequences $\varepsilon_n \rightarrow 0, p_n \rightarrow 2_s^*$ and $w_n \in \mathcal{N}_{p_n}$ such that

$$(4.2) \quad I_{p_n}(w_n) \leq m_{p_n} + \varepsilon_n \text{ and } \beta(w_n) \notin \Omega_r^+.$$

Then, by Proposition 1

$$(4.3) \quad I_{p_n}(w_n) \rightarrow m_*$$

and $\{w_n\}$ is bounded in $D_0^{s,2}(\Omega)$. Moreover if $|w_n|_{2_s^*} \rightarrow 0$ as $p_n \rightarrow 2_s^*$ we will have $I_{p_n}(w_n) = \frac{p_n-2}{2p_n} |w_n|_{2_s^*}^{2_s^*} \rightarrow 0$ which is a contradiction. Then, we are in a position to apply Lemma 6: let $t_n > 0$ be such that $t_n w_n \in \mathcal{N}_*$ and we suppose $t_n \rightarrow t_0 \in (0, 1]$. We evaluate

$$\begin{aligned} I_{p_n}(w_n) - I_*(t_n w_n) &= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|w_n\|^2 - \left(\frac{1}{2} - \frac{1}{2_s^*}\right) t_n^2 \|w_n\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|w_n\|^2 (1 - t_n^2) - \left(\frac{1}{p_n} - \frac{1}{2_s^*}\right) t_n^2 \|w_n\|^2 \\ &\geq o(1) \end{aligned}$$

which gives $m_* \leq I_*(t_n w_n) \leq I_{p_n}(w_n) + o(1)$. Then by (4.3), $I_*(t_n w_n) \rightarrow m_*$ for $n \rightarrow +\infty$. The Ekeland's variational principle implies that there exist $\{v_n\} \subset \mathcal{N}_*$ and $\{\mu_n\} \subset \mathbb{R}$ such that

$$\begin{aligned} \|t_n w_n - v_n\| &\rightarrow 0 \\ I_*(v_n) &= \frac{s}{N} \|v_n\|^2 \rightarrow m_* \\ I'_*(v_n) - \mu_n G'_*(v_n) &\rightarrow 0 \end{aligned}$$

and Lemma 3 ensures that $\{v_n\}$ is a PS sequence for the free functional I_* at level m_* . By the considerations made after Theorem 2,

$$v_n - U_{R_n}(\cdot - x_n) \rightarrow 0 \quad \text{in } D^{s,2}(\mathbb{R}^N),$$

where $\{x_n\} \subset \Omega, R_n \rightarrow +\infty$ and we can write $v_n = U_{R_n}(\cdot - x_n) + \zeta_n$ with a remainder ζ_n such that $\|\zeta_n\|_{D^{s,2}(\mathbb{R}^N)} \rightarrow 0$. It is clear that $t_n w_n = v_n + t_n w_n - \zeta_n$; so, denoting the remainder again with ζ_n , we have

$$t_n w_n = U_{R_n}(\cdot - x_n) + \zeta_n, \quad \text{with } \zeta_n \rightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Adapting the arguments given in [8, pags. 296-297] to the present case, after straightforward computations one show that

$$\beta(t_n w_n) = \beta(w_n) = x_n + o(1).$$

Since $x_n \in \Omega$, we get a contradiction with (4.2), and this concludes the proof. \square

4.2. Conclusion of the proof of Theorem 1. In the following, we add a subscript r ($r > 0$ and small as before) to denote the same quantities defined in the previous sections when the domain Ω is replaced by B_r ; namely integrals are taken on B_r and norms are taken for function spaces defined on the ball B_r . So

$$\mathcal{N}_{p,r} = \left\{ u \in D_0^{s,2}(B_r) : \|u\|_{D_0^{s,2}(B_r)}^2 = |u|_{L^p(B_r)}^p \right\}$$

and, for $u \in \mathcal{N}_{p,r}$ let

$$I_{p,r}(u) = \frac{p-2}{2p} \|u\|_{D_0^{s,2}(B_r)}^2.$$

It is easy to see (for example by means of the Palais Principle of Symmetric Criticality) that $\inf_{\mathcal{N}_{p,r}} I_{p,r}$ is achieved on a radially symmetric function $u_{p,r}$, then we have

$$m_{p,r} = \min_{\mathcal{N}_{p,r}} I_{p,r} = I_{p,r}(u_{p,r}).$$

Let

$$I_p^{m_{p,r}} = \{u \in \mathcal{N}_p : I_p(u) \leq m_{p,r}\}$$

be the sublevel of the functional I_p on the Nehari manifold (defined on the domain Ω); it is non vacuous since $m_p < m_{p,r}$.

For $p \in (2, 2_s^*)$ define the map $\Psi_{p,r} : \Omega_r^- \rightarrow \mathcal{N}_p$ such that

$$\Psi_{p,r}(y)(x) = \begin{cases} u_{p,r}(|x-y|) & \text{if } x \in B_r(y) \\ 0 & \text{if } x \in \Omega \setminus B_r(y) \end{cases}$$

and note that we have

$$\beta(\Psi_{p,r}(y)) = y \quad \text{and} \quad \Psi_{p,r}(y) \in I_p^{m_{p,r}}.$$

By Proposition 1 we can write $m_p + k_p = m_{p,r}$ where $k_p > 0$ and tends to zero if $p \rightarrow 2_s^*$. Then for $\varepsilon > 0$ provided by Proposition 2, there exists a $\bar{p} \in (2, 2_s^*)$ such that for every $p \in [\bar{p}, 2_s^*)$ it results $k_p < \varepsilon$, and if $u \in I_p^{m_{p,r}}$ we have

$$I_p(u) \leq m_{p,r} < m_p + \varepsilon.$$

Hence the following maps are well-defined (here h is the same as in (4.1)):

$$\Omega_r^- \xrightarrow{\Psi_{p,r}} I_p^{m_{p,r}} \xrightarrow{h \circ \beta} \Omega_r^-.$$

and the composite map $h \circ \beta \circ \Psi_{p,r}$ is homotopic to the identity of Ω_r^- . So, a well known property of the category, gives that the sublevel $I_p^{m_{p,r}}$ “dominates” the set Ω_r^- in the sense that

$$\text{cat}_{I_p^{m_{p,r}}} (I_p^{m_{p,r}}) \geq \text{cat}_{\Omega_r^-} (\Omega_r^-)$$

(see e.g. [6]) and our choice of r gives $\text{cat}_{\Omega_r^-} (\Omega_r^-) = \text{cat}_{\bar{\Omega}} (\bar{\Omega})$.

Summing up, we have found a sublevel of I_p on \mathcal{N}_p with category greater than $\text{cat}_{\bar{\Omega}} (\bar{\Omega})$. Since, as we have already said, the PS condition is verified on \mathcal{N}_p , applying the Lusternik-Schnirelmann theory we get the existence of at least $\text{cat}_{\bar{\Omega}} (\bar{\Omega})$ critical points for I_p on the manifold \mathcal{N}_p which give rise to solutions of (1.2).

In case of a non contractible domain Ω , the existence of another solution can be obtained with the same arguments of [2] (see also [8]). This is classical by now, but for completeness we recall the proof.

By assumption and by the choice of r it results $\text{cat}_{\Omega_r^+} (\Omega_r^-) > 1$, that is Ω_r^- is not contractible in Ω_r^+ .

If now the set $\Psi_{p,r}(\Omega_r^-)$ were contractible in $I_p^{m_{p,r}}$, then $\text{cat}_{I_p^{m_{p,r}}} (\Psi_{p,r}(\Omega_r^-)) = 1$ and this means that there exists a map $\mathcal{H} \in C([0, 1] \times \Psi_{p,r}(\Omega_r^-); I_p^{m_{p,r}})$ satisfying

$$\mathcal{H}(0, u) = u \quad \forall u \in \Psi_{p,r}(\Omega_r^-) \quad \text{and}$$

$$\exists w \in I_p^{m_{p,r}} : \mathcal{H}(1, u) = w \quad \forall u \in \Psi_{p,r}(\Omega_r^-).$$

Then $F = \Psi_{p,r}^{-1}(\Psi_{p,r}(\Omega_r^-))$ is closed, contains Ω_r^- and is contractible in Ω_r^+ as we can see by defining the map

$$\mathcal{G}(t, x) = \begin{cases} \beta(\Psi_{p,r}(x)) & \text{if } 0 \leq t \leq 1/2, \\ \beta(\mathcal{H}(2t - 1, \Psi_{p,r}(x))) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then also Ω_r^- would be contractible in Ω_r^+ giving a contradiction.

On the other hand we can choose a function $z \in \mathcal{N}_p \setminus \Psi_{p,r}(\Omega_r^-)$ so that the cone

$$\mathcal{C} = \{\theta z + (1 - \theta)u : u \in \Psi_{p,r}(\Omega_r^-), \theta \in [0, 1]\}$$

is compact and contractible in $D_0^{s,2}(\Omega)$ and $0 \notin \mathcal{C}$. Denoting with t_u the unique positive number provided by Lemma 2, it follows that if we set

$$\hat{\mathcal{C}} := \{t_u u : u \in \mathcal{C}\}, \quad M_p := \max_{\hat{\mathcal{C}}} I_p$$

then $\hat{\mathcal{C}}$ is contractible in $I_p^{M_p}$ and $M_p > m_{p,r}$. As a consequence also $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$.

Resuming, the set $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$ and not in $I_p^{m_{p,r}}$. Since the PS condition is satisfied we deduce the existence of another critical point with critical level between $m_{p,r}$ and M_p .

That the solutions we have found are positive, is a simple consequence of the fact that we can apply all the previous machinery replacing the functional (2.1) with

$$I_p(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} u^2 - \frac{1}{p} \int_{\mathbb{R}^N} (u^+)^p, \quad u \in D_0^{s,2}(\Omega)$$

and then by the Maximum Principle we conclude.

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